16th Int. Conf. on Principles and Practice of Constraint Programming

St Andrews, Scotland,6-10th September 2010

Structural Tractability of Enumerating CSP Solutions

UNIVERSITÀ DELLA CALABRIA
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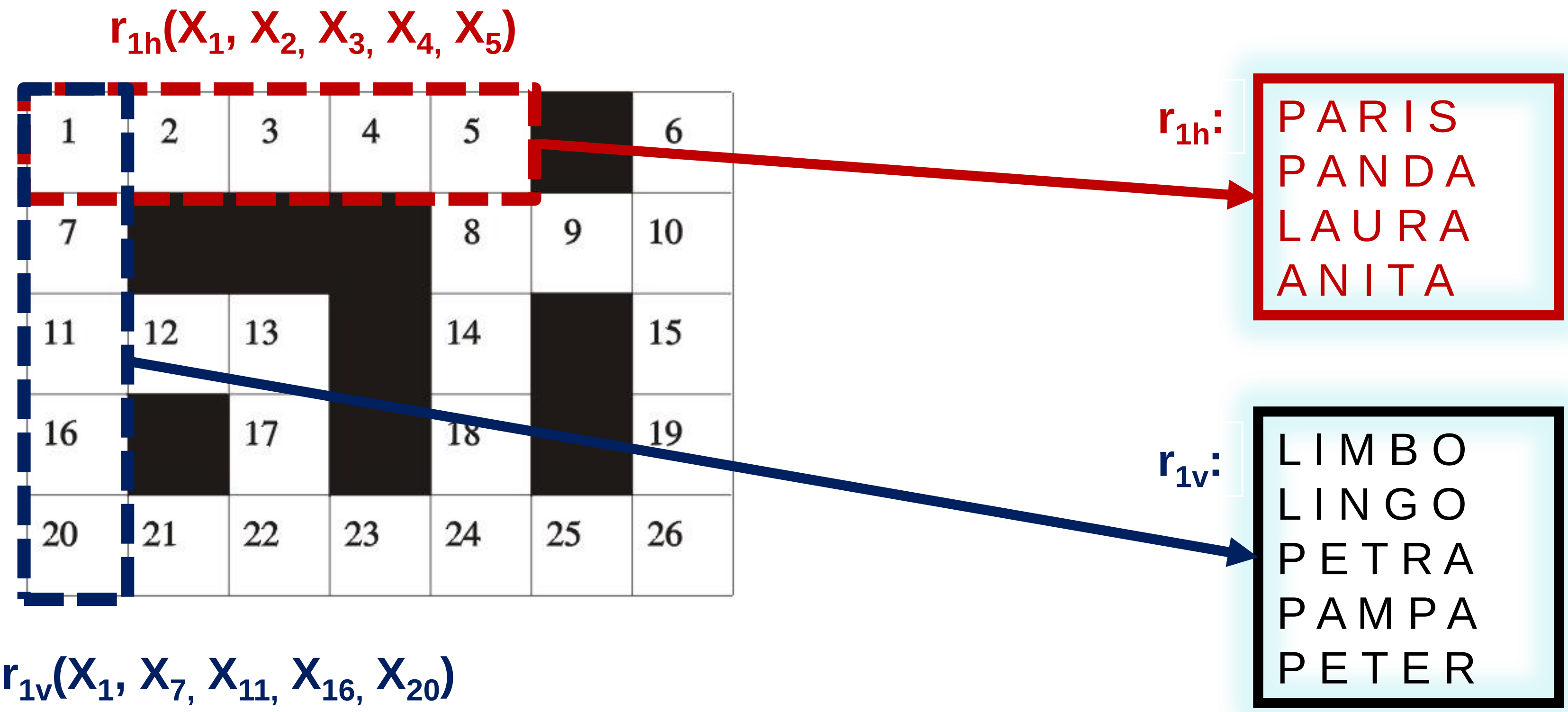
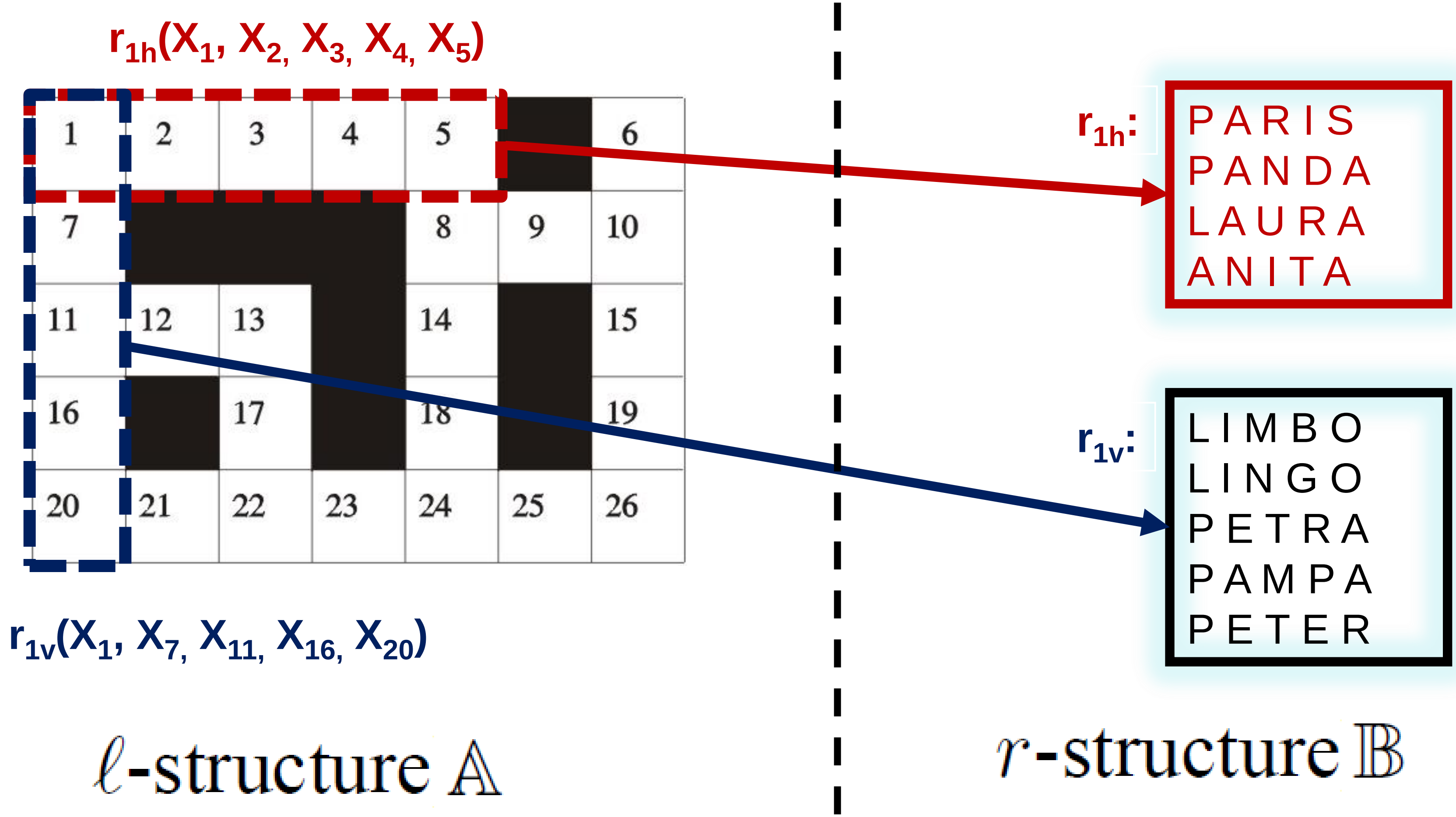
CSPs as Homomorphism Problems



- Set of variables $\{X_1,\ldots,X_{26}\}$
- Set of constraint scopes

- Set of constraint relations

CSPs as Homomorphism Problems



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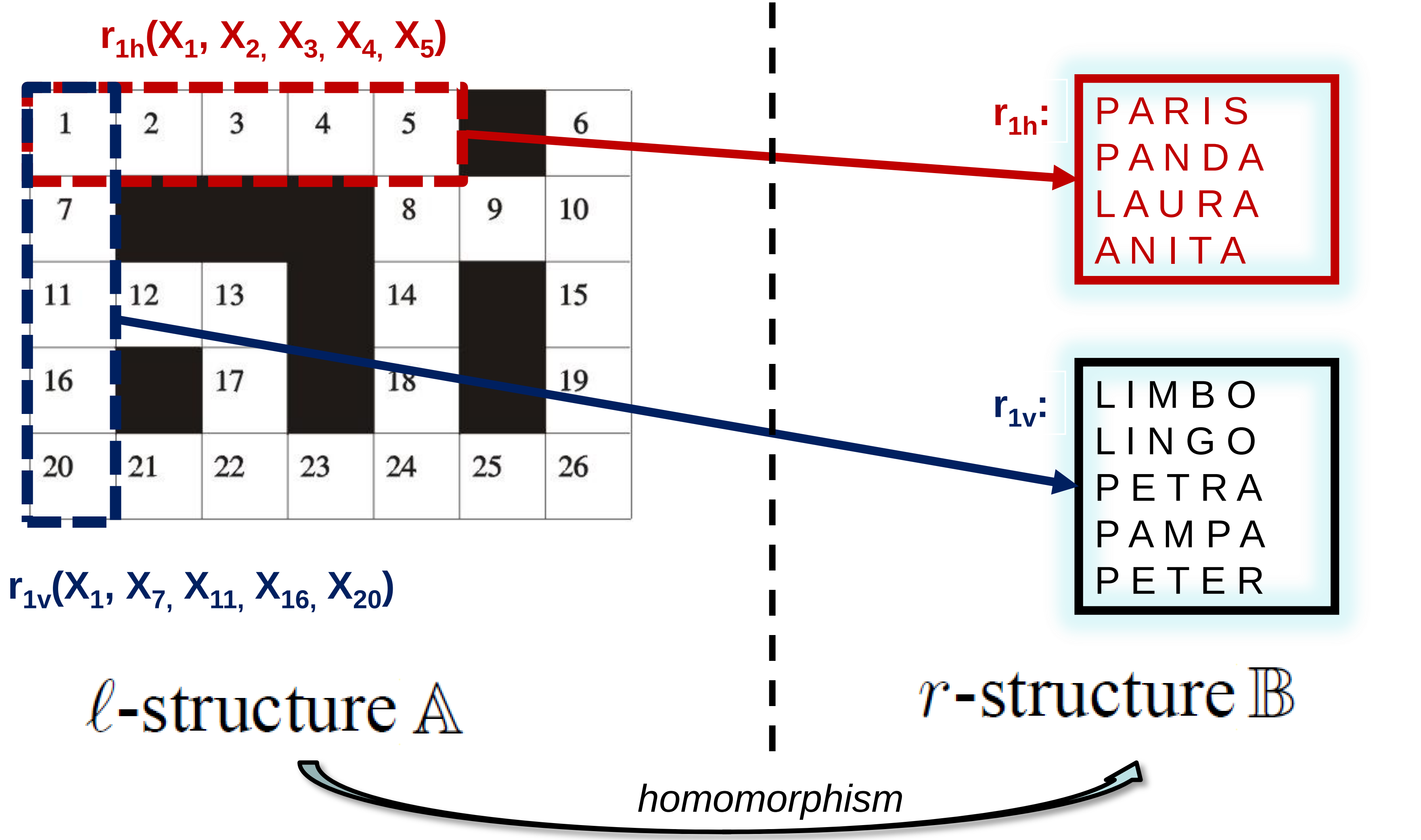

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INPUT: CSP instance $(\mathbb{A}, \mathbb{B})$

- Decide the *existence* of a homomorphism
- *Enumerate* all the homomorphisms $\mathbb{A}^{\mathbb{B}}$
- For a set of variales X, enumerate the *projection* $\mathbb{A}^{\mathbb{B}}[X]$

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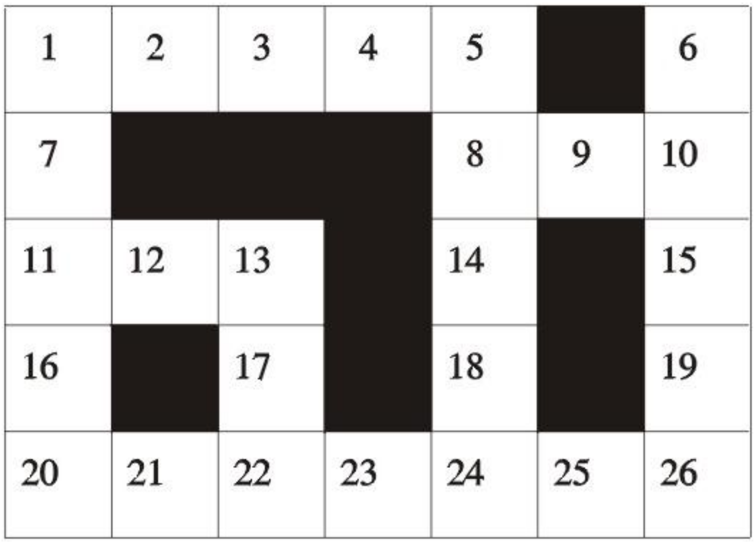
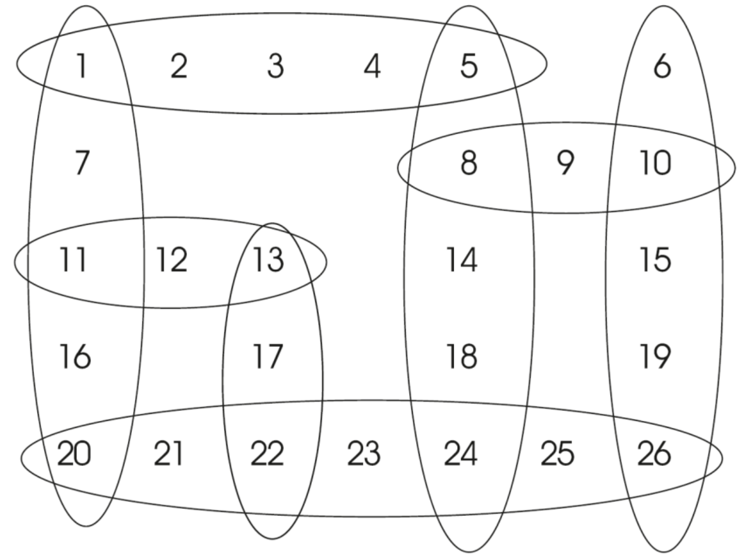
- Tractable decision and closure properties imply tractable search
  [R. Dechter and A. Itai, 1992]
- Non-uniform case
  [D. Cohen, 2004]

CSPs and Hypergraphs



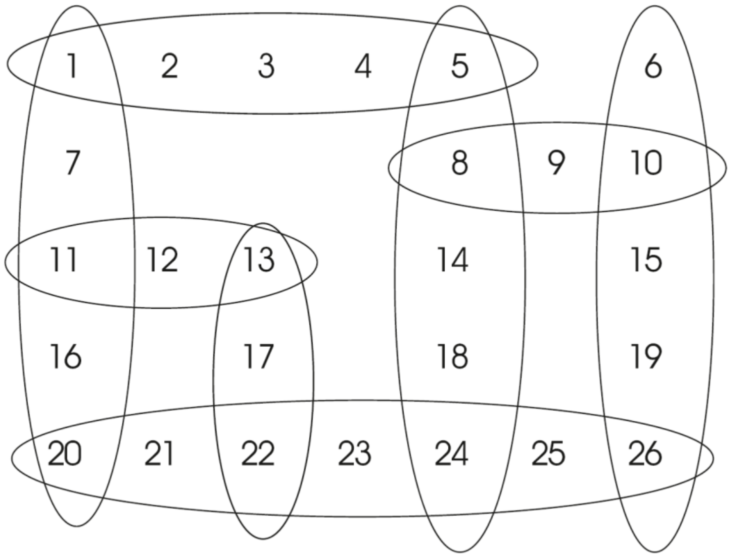
ℓ-structure $\mathbb{A}$



$\mathcal{H}_{\mathbb{A}}$

- Variables map to nodes
- Scopes map to hyperedges

Structurally Restricted CSPs



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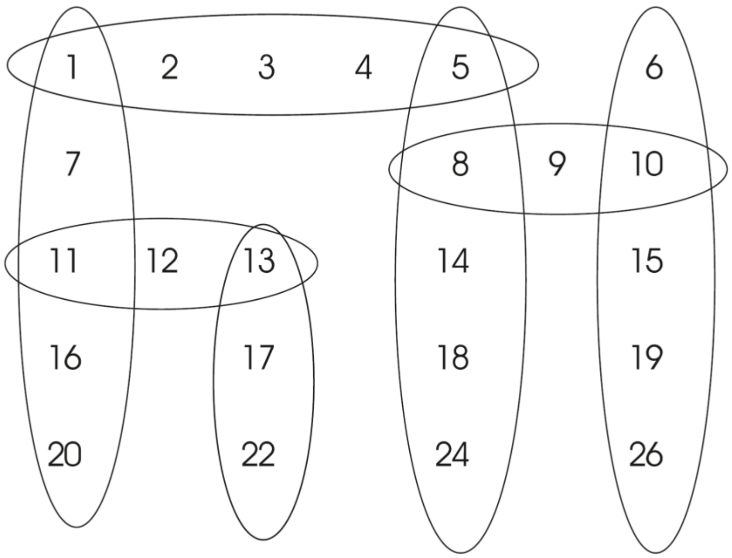

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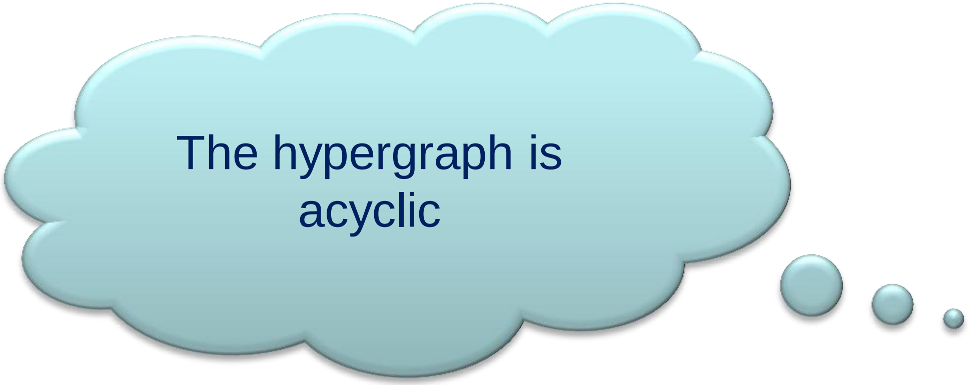
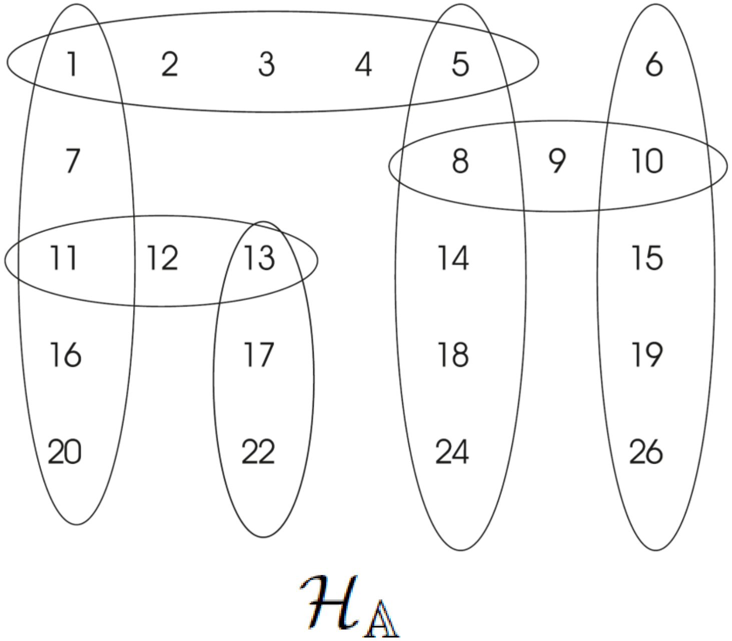
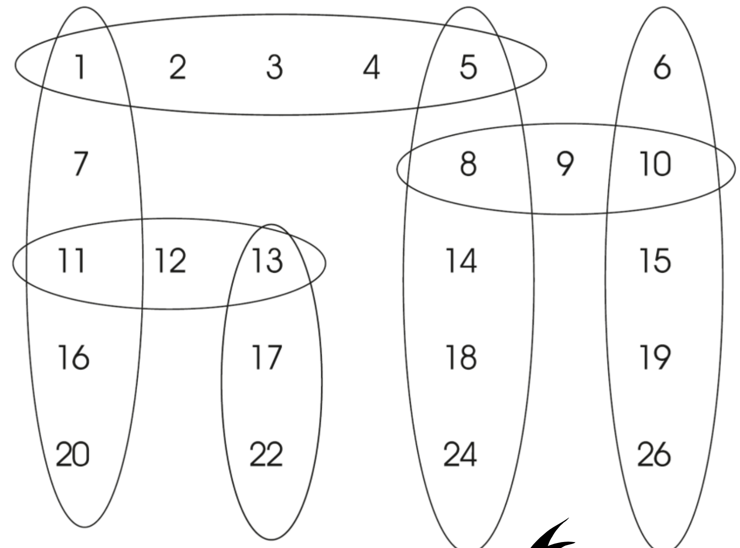




- Acyclicity is efficiently recognizable
- Acyclic CSPs can be efficiently solved
- Generalized arc consistency $\rightarrow$ Global consistency

Structurally Restricted CSPs

The hypergraph is acyclic



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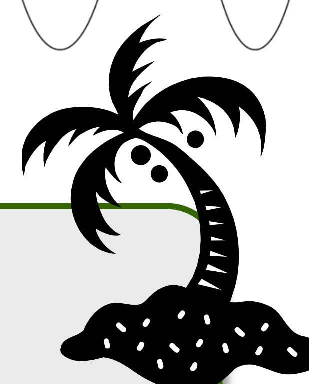

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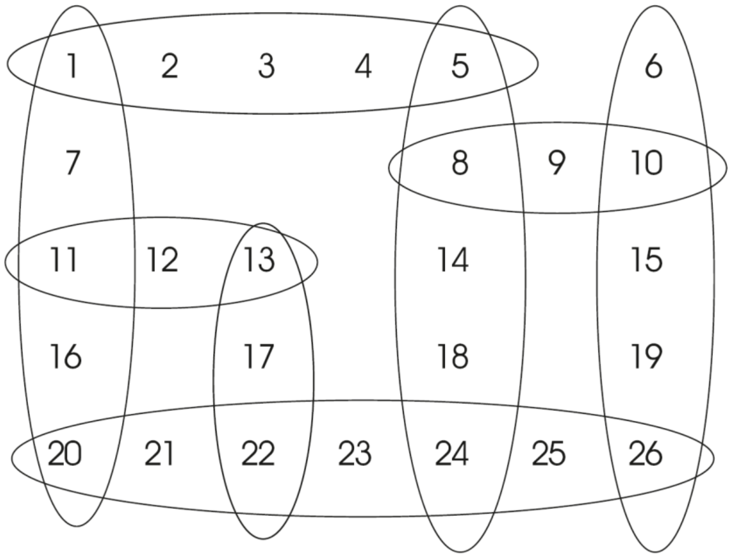
Decomposition Methods



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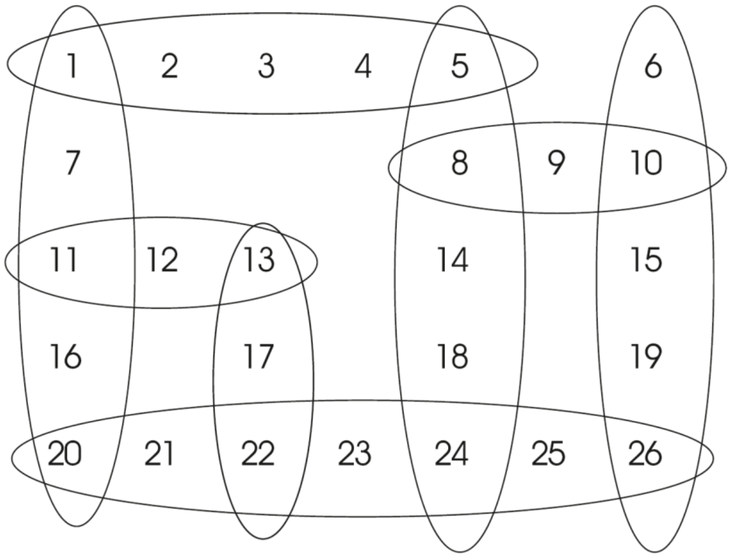

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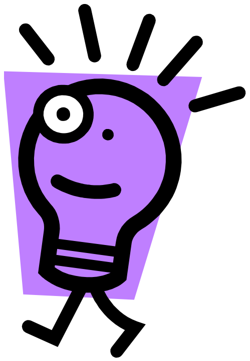
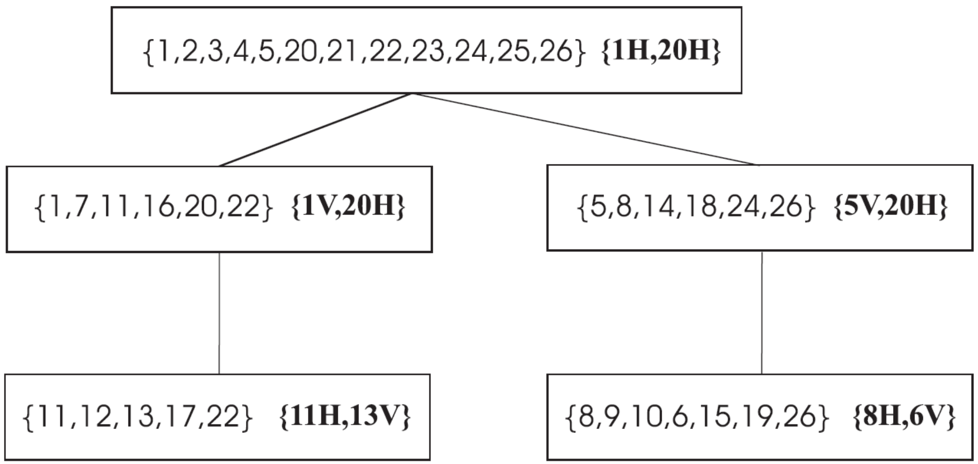
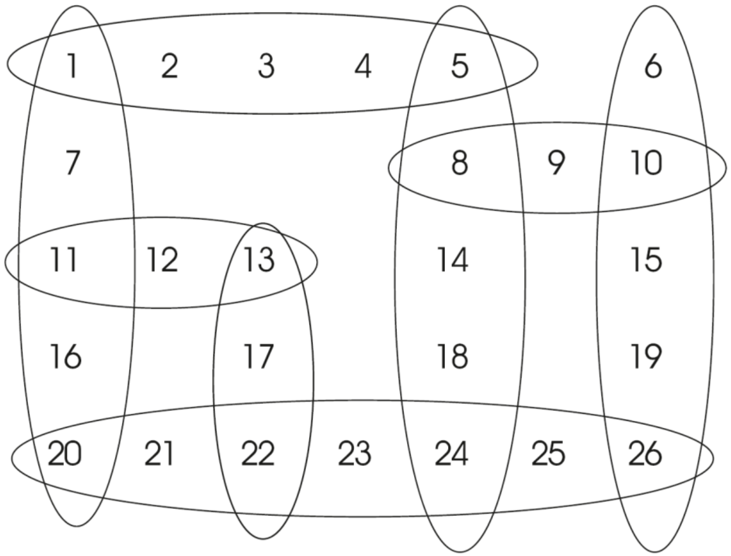


Transform the hypergraph into an acyclic one:

- Organize its edges (or nodes) in clusters
- Arrange the clusters as a tree,
  by satisfying the connectedness condition

Generalized Hypertree Decompositions





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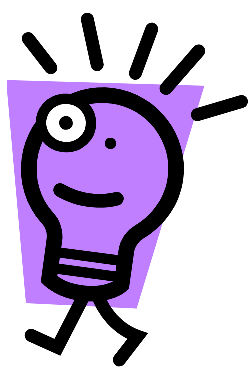

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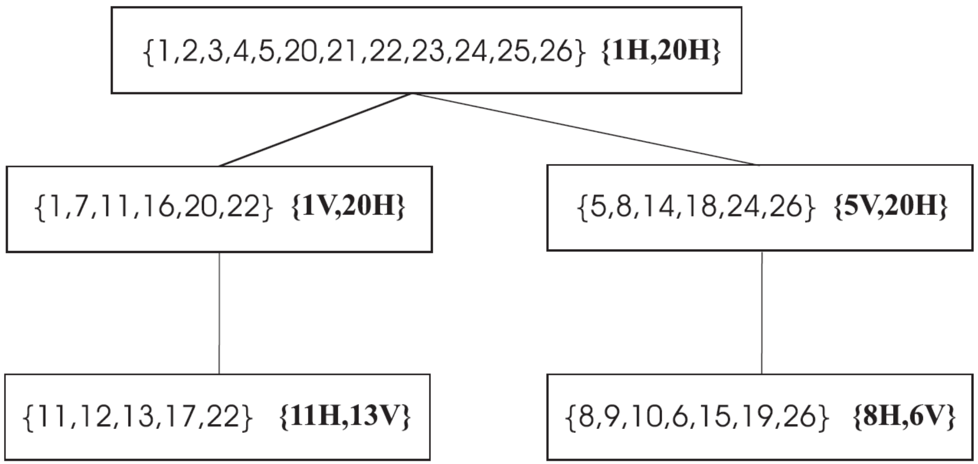
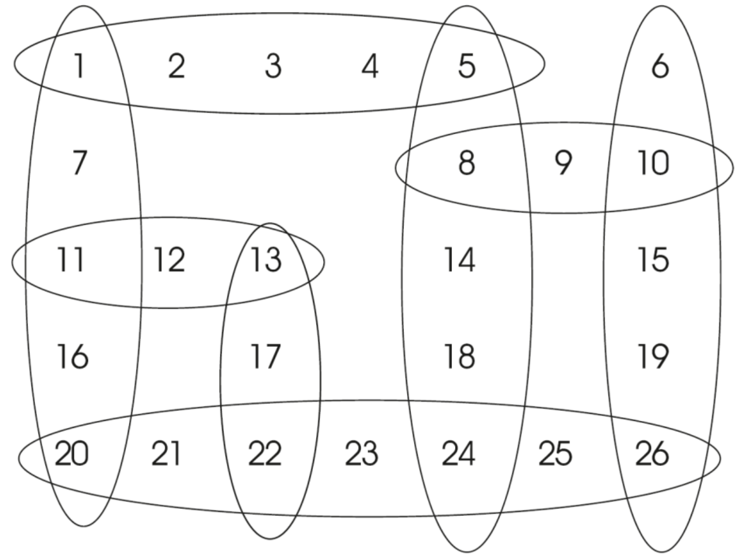
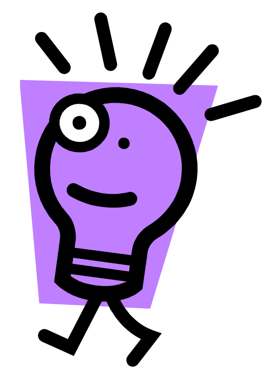




$\mathcal{H}_{\mathbb{A}}$

Each cluster can be seen as a **subproblem**



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Outline

Decomposition Methods and Tree Projections

Enumeration without Certificates

Enumeration with Certificates

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- Enumeration with Certificates

Revisiting Decomposition Methods

CSP instance $(\mathbb{A}, \mathbb{B})$

$$\mathbb{A}_{\mathcal{V}} = \ell\text{-DM}(\mathbb{A}) \quad \mathbb{B}_{\mathcal{V}} = r\text{-DM}(\mathbb{A}, \mathbb{B})$$

Scopes

Solutions

Work on subproblems

Revisiting Decomposition Methods

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(Noticeable) Examples

CSP instance $(\mathbb{A}, \mathbb{B})$

$$\mathbb{A}_{\mathcal{V}} = \ell\text{-DM}(\mathbb{A}) \quad \mathbb{B}_{\mathcal{V}} = r\text{-DM}(\mathbb{A}, \mathbb{B})$$

- *Treewidth*: take all views that can be computed with at most k variables
- *Generalized hypertree width*: take all views that can be computed by joining at most k atoms (k query views)
- *Fractional hypertree width*: take all views that can be computed through subproblems having fractional cover at most k (or use Marx's $O(k^3)$ approximation to have polynomially many views)

Acyclicity in Decomposition Methods

CSP instance $(\mathbb{A}, \mathbb{B})$

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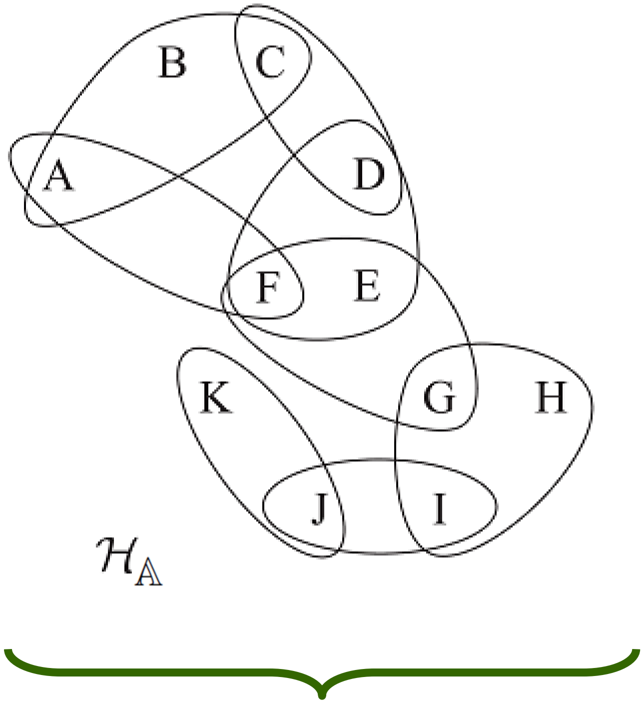
Working on subproblems is not necessarily beneficial…

Tree Projections (by Example)

$\mathbb{A}$: $r_1(A,B,C)$ $r_2(A,F)$ $r_3(C,D)$ $r_4(D,E,F)$
$r_5(E,F,G)$ $r_6(G,H,I)$ $r_7(I,J)$ $r_8(J,K)$



Structure of the CSP

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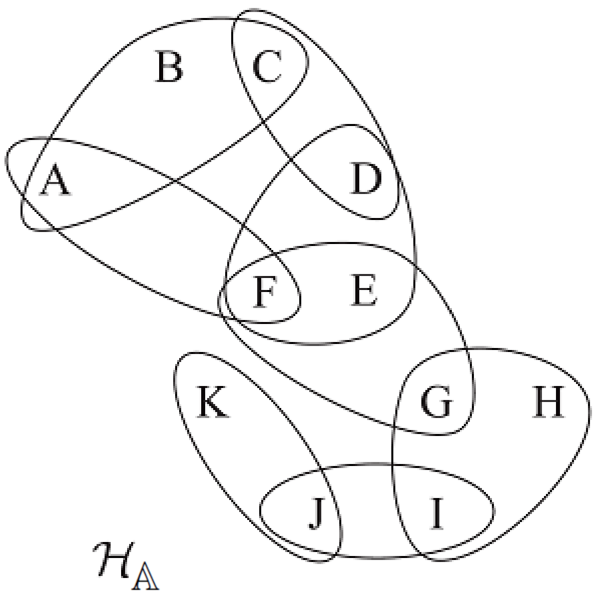
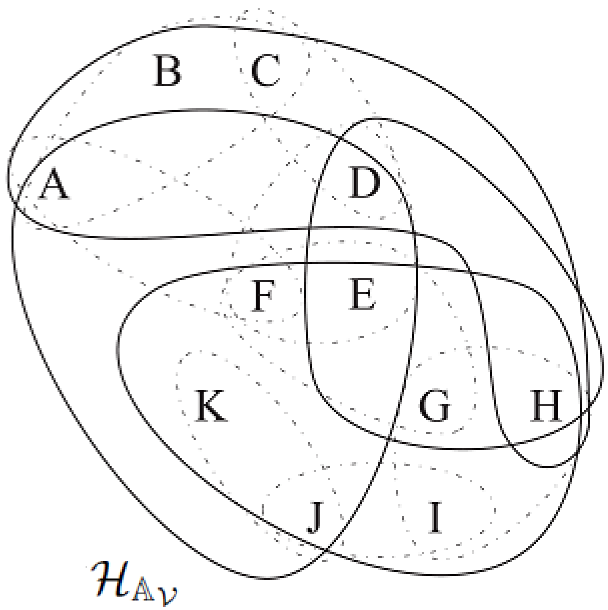


Structure of the CSP



Available Views

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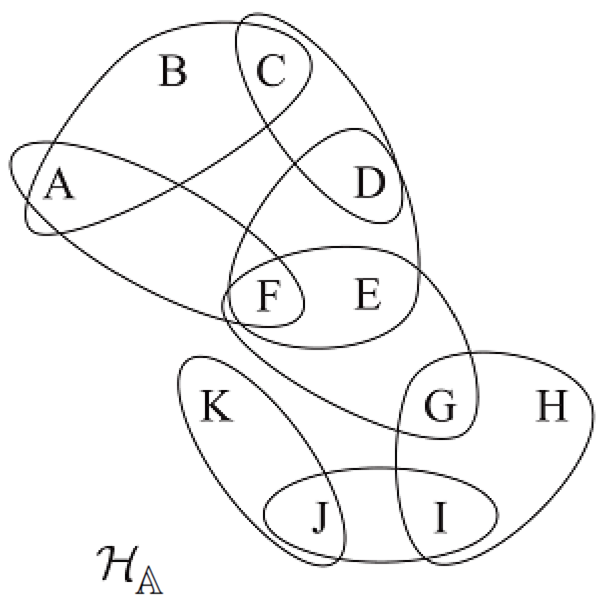

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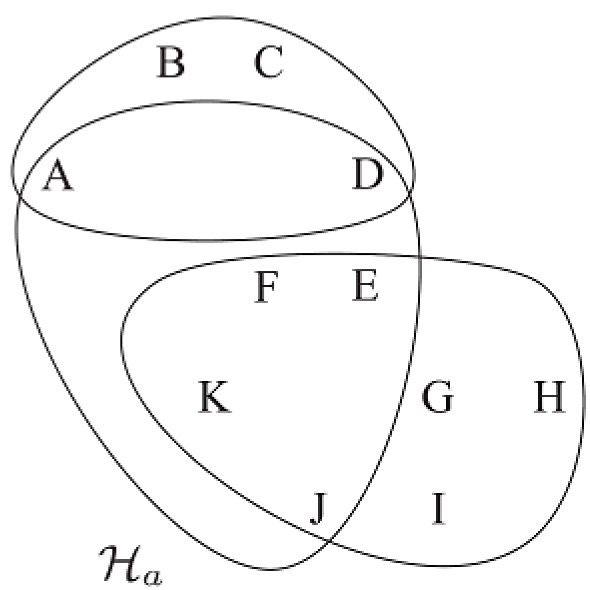
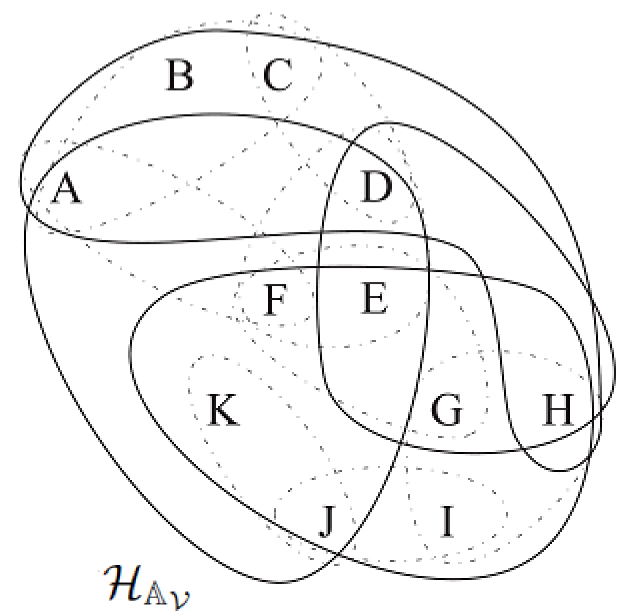


Tree Projection



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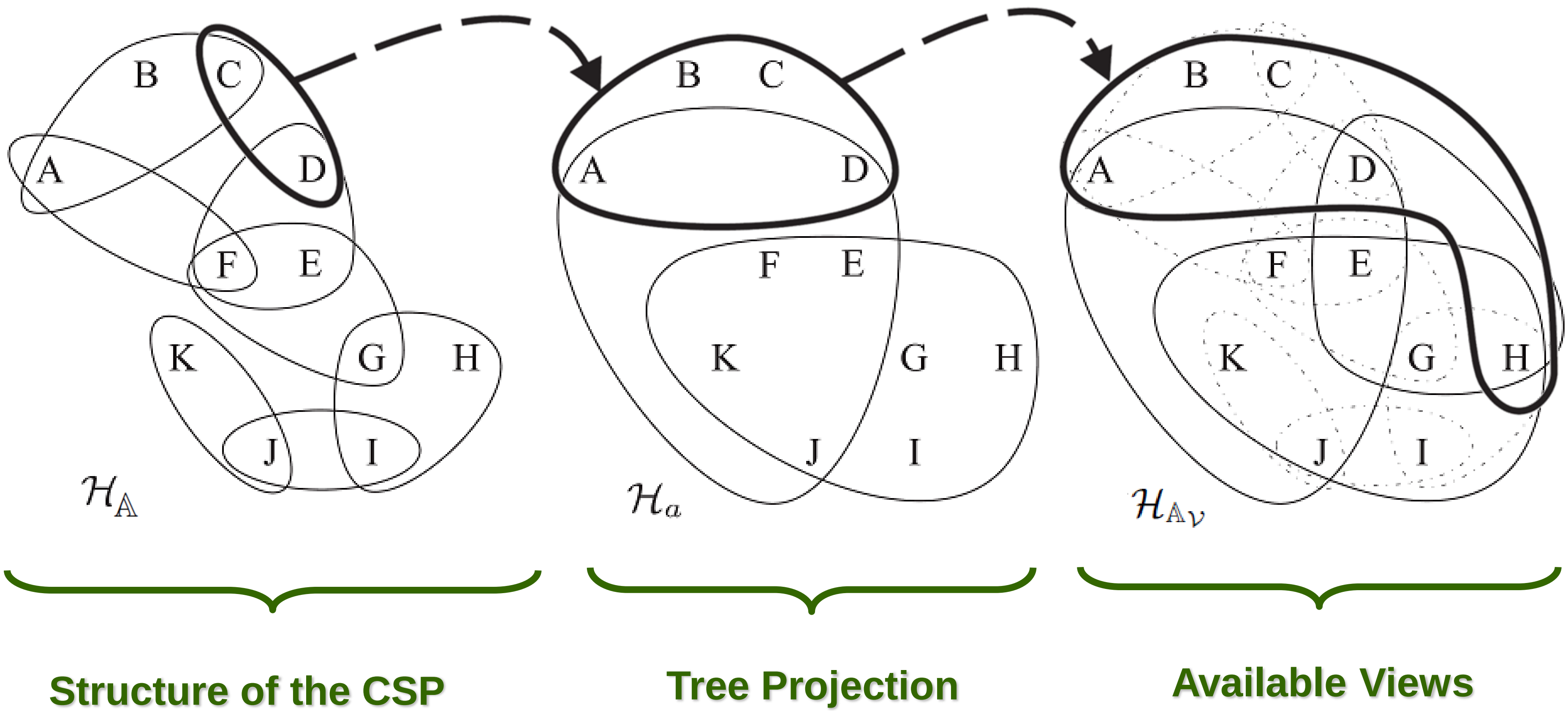
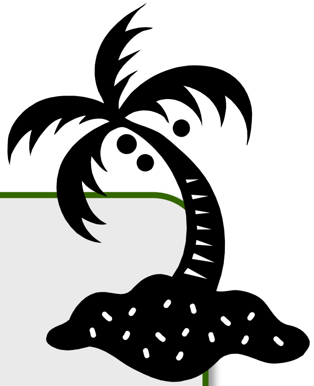


From Acyclicity to Tree Projections

- Deciding whether a tree projection exists is NP-complete
  [G. Gottlob, Z. Miklós, and T. Schwentick, 2009]
- The existence of a tree projection ensures polynomial-time solvability
  [N. Goodman and O. Shmueli, 1984], [Y. Sagiv and O. Shmueli, 1993]



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The Algorithm: Backtracking and Propagation

Input: An ECSP instance $(\mathbb{A}, \mathbb{B}, O)$, where $O = \{X_1, \ldots, X_m\}$;
Output: $\mathbb{A}^{\mathbb{B}}[O]$;
Method: update $(\mathbb{A}, \mathbb{B}, O)$ with any of its domain-restricted versions;
let $\mathbb{A}_{\mathcal{V}} := \ell\text{-}\mathtt{DM}(\mathbb{A})$, $\mathbb{B}_{\mathcal{V}} := r\text{-}\mathtt{DM}(\mathbb{A}, \mathbb{B})$;
invoke $\mathtt{Propagate}(1, (\mathbb{A}_{\mathcal{V}}, \mathbb{B}_{\mathcal{V}}), m, \langle\rangle)$;

Procedure $\mathtt{Propagate}(i$: integer, $(\mathbb{A}_{\mathcal{V}}, \mathbb{B}_{\mathcal{V}})$: pair of structures, m: integer, $\langle a_1, \ldots, a_{i-1}\rangle$: tuple of values in A^i);
begin
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for each element $\langle a_i \rangle \in activeValues$ **do**
  if $i = m$ **then**
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The solution is
not certified!

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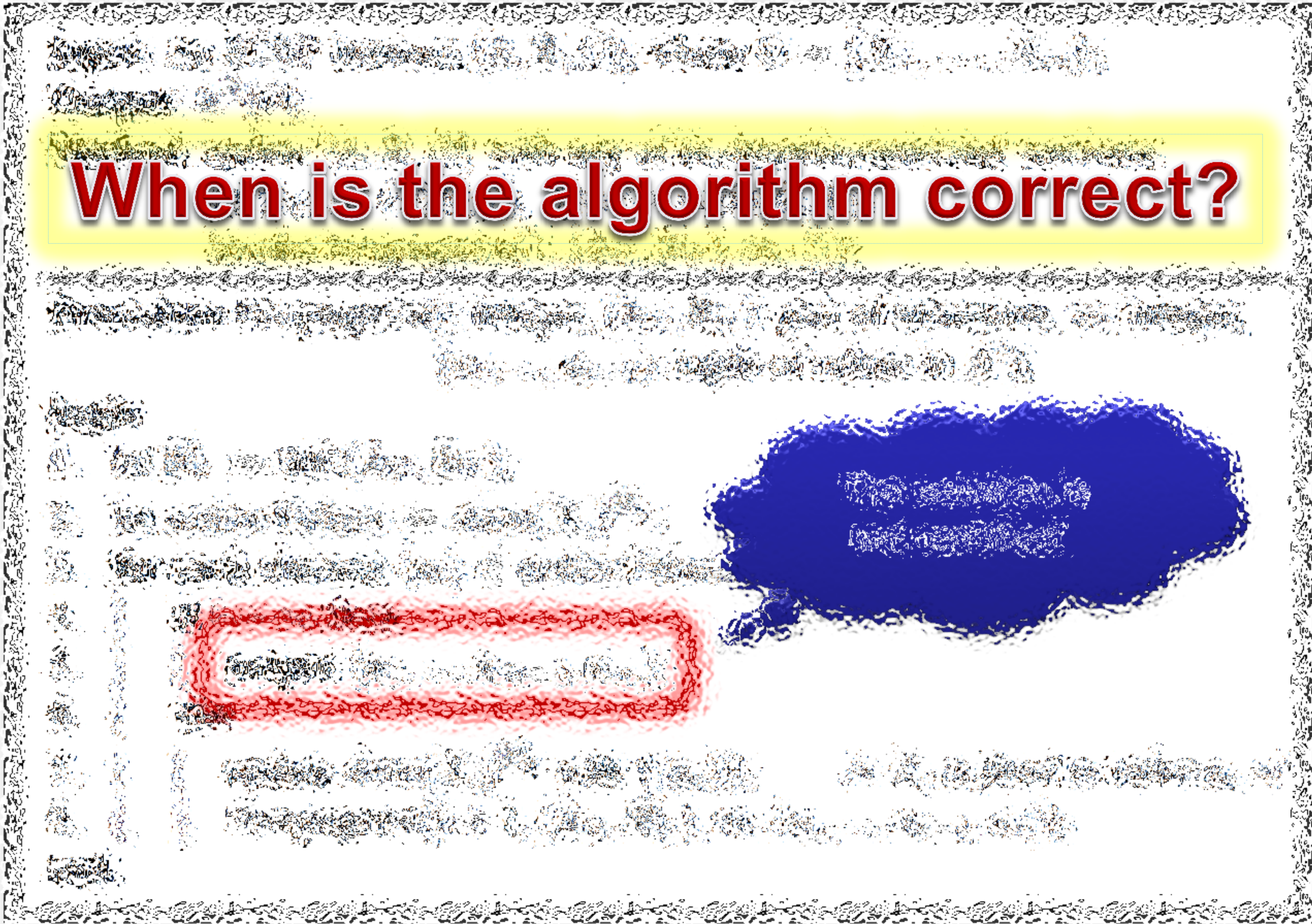
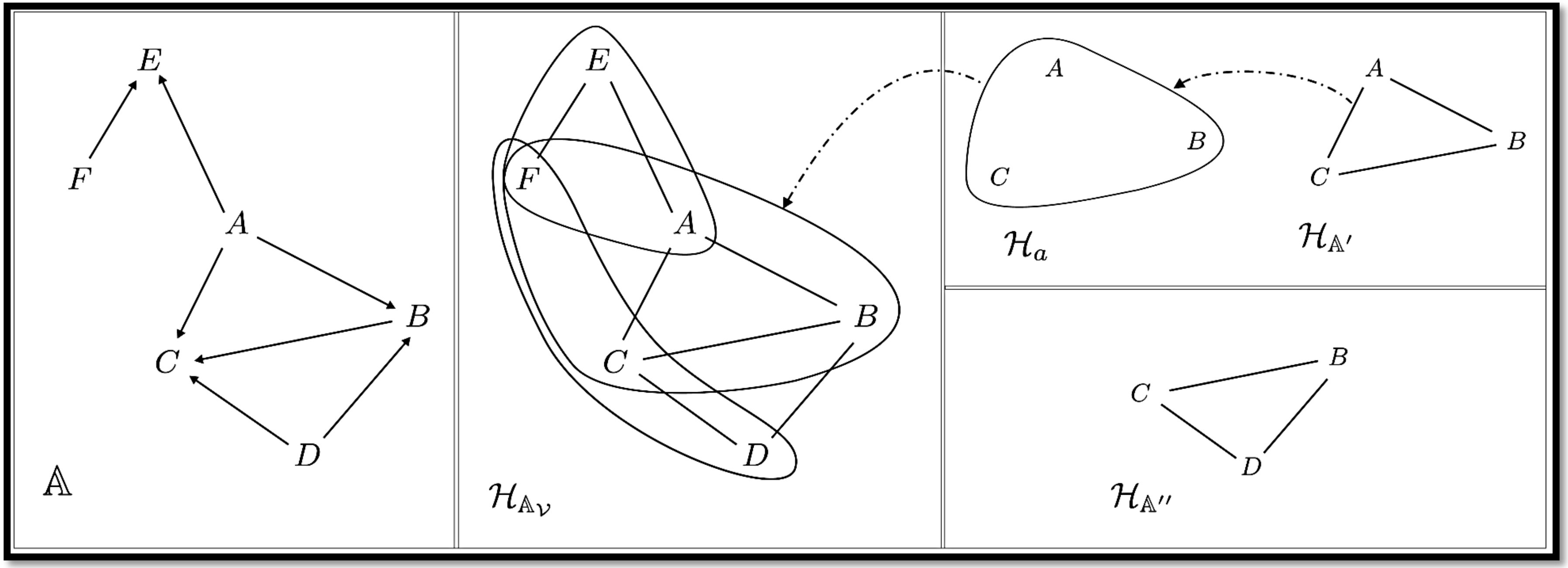


Tp-covering



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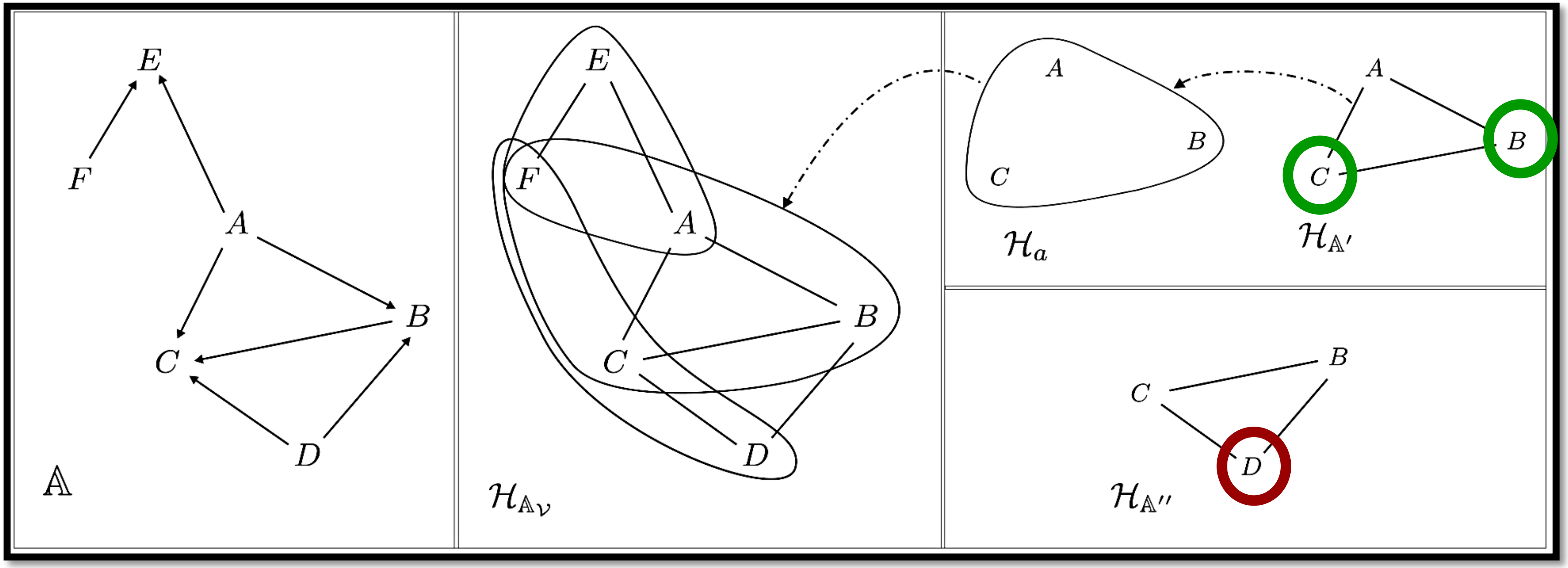
$\mathcal{H}_{\mathbb{A}''}$

Tp-covering



{B,C} is individually tp-covered

{D} is not individually tp-covered

Tight Characterizations for the Correctness

When is the algorithm correct?

Thm.

Let $\mathbb{A}$ be an ℓ-structure, and let $O \subseteq A$ be a set of variables. The following are equivalent:

(1) O is individually tp-covered

(2) For every r-structure $\mathbb{B}$, $\texttt{ComputeAllSolutions}_{\mathrm{DM}}$ computes $\mathbb{A}^{\mathbb{B}}[O]$.

Proof: Discussion on the Base (Decision) Case

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Over the views, i.e., k-consistency

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Proof: Discussion on the Base (Decision) Case

Thm [G. Greco and F. Scarcello, 2010].

Let $\mathbb{A}$ be an ℓ-structure, and let $\mathbb{A}_{\mathcal{V}}$ be a v-structure.
The following are equivalent:

(1) There is a core $\mathbb{A}'$ of $\mathbb{A}$ such that $(\mathcal{H}_{\mathbb{A}'}, \mathcal{H}_{\mathbb{A}_{\mathcal{V}}})$ has a tree projection

(2) For every r-structure $\mathbb{B}$, for every r-structure $\mathbb{B}_{\mathcal{V}}$ that is legal, enforcing generalized arc consistency on $\mathbb{B}_{\mathcal{V}}$ is a correct decision procedure

- For *generalized hypertree decompositions*, the existence of a tree projection for the core is also necessary to imply "GAC $\rightarrow$ Global" [G. Greco and F. Scarcello, 2010]

Complexity Issues of $\texttt{ComputeAllSolutions}_{\mathrm{DM}}$

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Assume $FPT \neq W[1]$. Let $\mathbf{A}$ be any class of ℓ-structures of bounded arity. Then, the following are equivalent:

(1) $\mathbf{A}$ *has bounded treewdith modulo homomorphic equivalence;*
(2) For every $\mathbb{A} \in \mathbf{A}$, for every r-structure $\mathbb{B}$, and for every set of variables $O \subseteq drv(\mathbb{A})$, the ECSP *instance $(\mathbb{A}, \mathbb{B}, O)$ is solvable WPD.*

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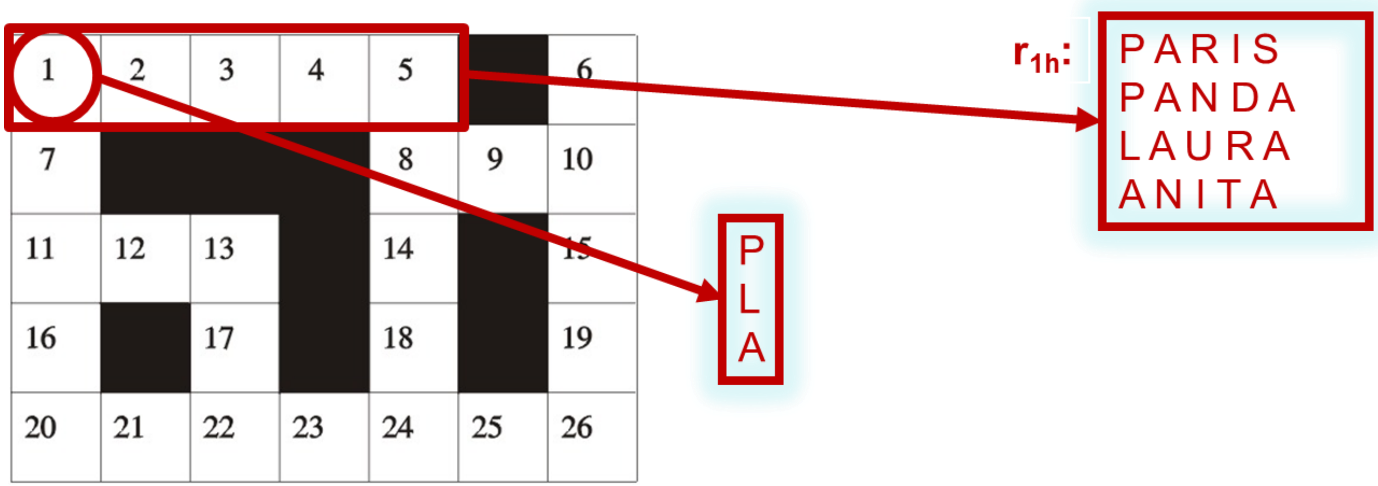

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There is no efficient algorithm for the no-promise problem

Outline

Decomposition Methods and Tree Projections

Enumeration without Certificates

Enumeration with Certificates

A Simple Modification

Input: An ECSP instance $(\mathbb{A}, \mathbb{B}, O)$, where $O = \{X_1, \ldots, X_m\}$;
Output: for each solution $h \in \mathbb{A}^{\mathbb{B}}[O]$, a certified solution (h, h');
Method: let $A = \{X_1, \ldots, X_m, X_{m+1}, \ldots, X_n\}$ be the variables of $\mathbb{A}$;
update $(\mathbb{A}, \mathbb{B}, A)$ with any of its domain restricted versions;
let $\mathbb{A}_{\mathcal{V}} := \ell\text{-DM}(\mathbb{A})$, $\mathbb{B}_{\mathcal{V}} := r\text{-DM}(\mathbb{A}, \mathbb{B})$;
invoke $\texttt{CPropagate}(1, (\mathbb{A}_{\mathcal{V}}, \mathbb{B}_{\mathcal{V}}), m, \langle\rangle)$;

Procedure $\texttt{CPropagate}(i$: integer, $(\mathbb{A}_{\mathcal{V}}, \mathbb{B}_{\mathcal{V}})$: pair of structures, m: integer,
$\langle a_1, \ldots, a_{i-1}\rangle$: tuple of values in A^i);

begin
let $\mathbb{B}'_{\mathcal{V}} := \texttt{GAC}(\mathbb{A}_{\mathcal{V}}, \mathbb{B}_{\mathcal{V}})$;
if $i > 1$ and $\mathbb{B}'_{\mathcal{V}}$ is empty **then** output "DM failure" and HALT;
let $activeValues := dom(X_i)^{\mathbb{B}'_{\mathcal{V}}}$;
for each element $\langle a_i \rangle \in activeValues$ **do**
  if $i = n$ **then**
    output the certified solution $(\langle a_1, \ldots, a_m\rangle, \langle a_{m+1}, \ldots, a_n\rangle)$;
  else
    update $dom(X_i)^{\mathbb{B}'_{\mathcal{V}}}$ with $\{\langle a_i \rangle\}$;   /* X_i *is fixed to value* a_i */
    $\texttt{CPropagate}(i+1, (\mathbb{A}_{\mathcal{V}}, \mathbb{B}'_{\mathcal{V}}), m, \langle a_1, \ldots, a_{i-1}, a_i\rangle)$;
    if $i > m$ **then** BREAK;
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Thm.

Let $\mathbb{A}$ *be an* ℓ*-structure, and* $O \subseteq A$ *be a set of variables. Then, for every* r*-structure* $\mathbb{B}$, $\texttt{ComputeCertifiedSolutions}_{\mathrm{DM}}$ *computes WPD a subset of the solutions in* $\mathbb{A}^{\mathbb{B}}[O]$*, with a certificate for each of them. Moreover,*

- *If* $\texttt{ComputeCertifiedSolutions}_{\mathrm{DM}}$ *outputs "*DM *failure", then* $(\mathcal{H}_{\mathbb{A}}, \mathcal{H}_{\ell\text{-}\mathrm{DM}(\mathbb{A})})$ *does not have a tree projection;*
- *otherwise,* $\texttt{ComputeCertifiedSolutions}_{\mathrm{DM}}$ *computes WPD* $\mathbb{A}^{\mathbb{B}}[O]$.

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Assume FPT $\neq$ W[1]. *Let* **A** *be any bounded-arity recursively-enumerable class of* ℓ*-structures closed under taking minors. Then, the following are equivalent:*

(1) **A** *has bounded treewdith;*

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Comparing The Results

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Assume $FPT \neq W[1]$. Let $\mathbf{A}$ be any class of ℓ-structures of bounded arity. Then, the following are equivalent:

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Thank you!